Discrete Mathematics

Relation and their properties

Chapter 9

Rosen

9.3 Representing relations

- Can use ordered set, graph to represent sets
- Generally, matrices are better choice
- Suppose that R is a relation from $A=\{a_1, a_2, \ldots, a_m\}$ to $B=\{b_1, b_2, \ldots, b_n\}$. The relation R can be represented by the matrix $M_R=[m_{ij}]$ where

  $m_{ij}=1$ if $(a_i,b_j) \in R$,

  $m_{ij}=0$ if $(a_i,b_j) \notin R$,
- A zero-one (binary) matrix

Example

- Suppose that $A=\{1,2,3\}$ and $B=\{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1=1$, $a_2=2$, and $a_3=3$, and $b_1=1$, and $b_2=2$
- As $R=\{(2,1), (3,1), (3,2)\}$, the matrix R is

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

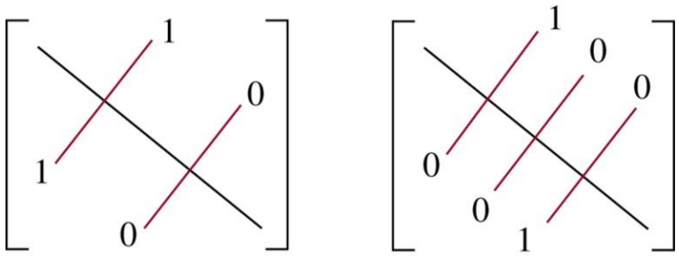
Matrix and relation properties

- The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties
- Recall that a relation R on A is reflexive if $(a,a) \in R$. Thus R is reflexive if and only if $(a_i,a_i) \in R$ for $i=1,2,\ldots,n$
- Hence R is reflexive iff $m_{ii}=1$, for $i=1,2,\ldots, n$.
- R is reflexive if *all* the elements on the main diagonal of M_R are 1

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

Symmetric

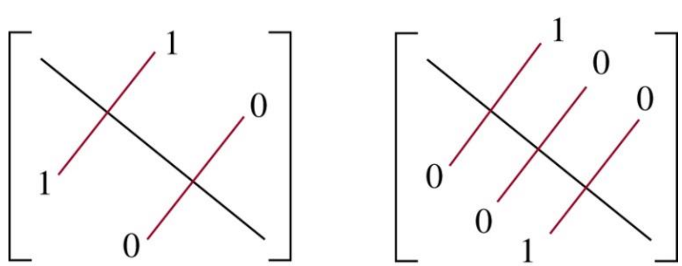
- The relation R is symmetric if $(a,b) \in R$ implies that $(b,a) \in R$
- In terms of matrix, R is symmetric if and only $m_{ji}=1$ whenever $m_{ij}=1$, i.e., $M_R=(M_R)^T$
- R is symmetric iff M_R is a symmetric matrix



(a) Symmetric (b) Antisymmetric

Antisymmetric

- The relation R is symmetric if (a,b)∈R and (b,a)∈R imply a=b
- The matrix of an antisymmetric relation has the property that if $m_{ij}=1$ with $i \neq j$, then $m_{ji}=0$
- In other words, either $m_{ij}=0$ or $m_{ji}=0$ when $i \neq j$



(a) Symmetric (b) Antisymmetric

Example

- Suppose that the relation R on a set is represented by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

  Is R reflexive, symmetric or antisymmetric?

- As all the diagonal elements are 1, R is reflexive. As M_R is symmetric, R is symmetric. It is also easy to see R is not antisymmetric

Union, intersection of relations

- Suppose R1 and R2 are relations on a set A represented by M_{R1} and M_{R2}
- The matrices representing the union and intersection of these relations are

$$M_{R1 \cup R2} = M_{R1} \vee M_{R2}$$

$$M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$$

Example

- Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices for $R_1 \cup R_2$ and $R_1 \cap R_2$?

$$M_{R_1 \cup R2} = M_{R_1} \vee M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad M_{R_1 \cap R2} = M_{R_1} \wedge M_{R2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Composite of relations

- Suppose R is a relation from A to B and S is a relation from B to C. Suppose that A, B, and C have m, n, and p elements with M_S, M_R
- Use Boolean product of matrices
- Let the zero-one matrices for $S \circ R$, R, and S be $M_{S \circ R}=[t_{ij}]$, $M_R=[r_{ij}]$, and $M_S=[s_{ij}]$ (these matrices have sizes $m \times p$, $m \times n$, $n \times p$)
- The ordered pair $(a_i, c_j) \in S \circ R$ iff there is an element b_k s.t.. $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$
- It follows that $t_{ij}=1$ iff $r_{ik}=s_{kj}=1$ for some k, $M_{S \circ R} = M_R \odot M_S$

Boolean product (Section 3.8)

- Boolean product A ⊙ B is defined as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Replace x with ∧, and + with ∨

$$A \cdot B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Boolean power (Section 3.8)

- Let A be a square zero-one matrix and let r be positive integer. The r-th Boolean power of A is the Boolean product of r factors of A, denoted by $A^{[r]}$
- $A^{[r]} = \underbrace{A \odot A \odot A \dots \odot A}_{\text{r times}}$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^{[2]} = A \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{[3]} = A^{[2]} \cdot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A^{[4]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example

- Find the matrix representation of S∘R

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{S \cdot R} = M_R \cdot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Powers R^n

- For powers of a relation

$$M_{R^n} = M_R^{[n]}$$

- The matrix for R^2 is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

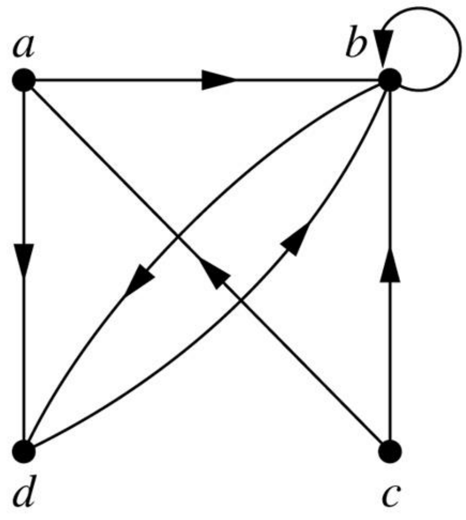
$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing relations using digraphs

- A **directed** graph, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**)
- The vertex a is called the **initial** vertex of the edge (a,b), and vertex b is called the **terminal** vertex of the edge
- An edge of the form (a,a) is called a **loop**

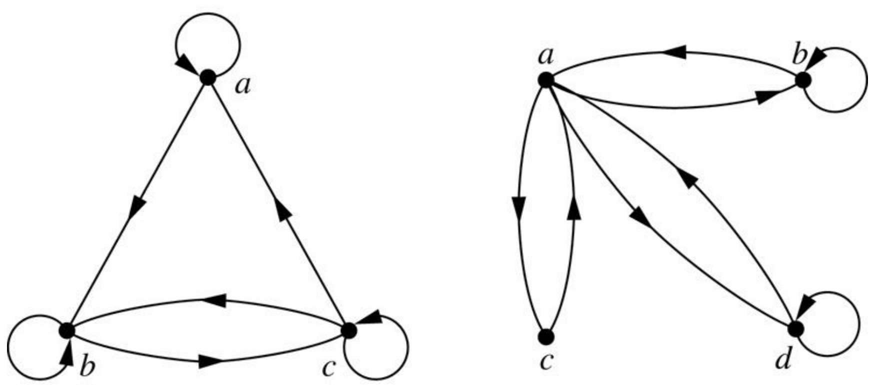
Example

- The directed graph with vertices a, b, c, and d, and edges (a,b), (a,d), (b,b), (b,d), (c,a), (c,b), and (d,b) is shown



$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example



(a) Directed graph of R (b) Directed graph of S

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad M_S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

- R is reflexive. R is neither symmetric (e.g., (a,b)) nor antisymmetric (e.g., (b,c), (c,b)). R is not transitive (e.g., (a,b), (b,c))
- S is not reflexive. S is symmetric but not antisymmetric (e.g., (a,c), (c,a)). S is not transitive (e.g., (c,a), (a,b))